INFINITY IN YOUR HAND (2-9)

Students in **grades 2-4** shade grids (or use link cubes) to build area (or volume) representations of unit fractions 1/2, 1/4, 1/8, and so on, and notice that they all fit in a single square (or square grid of link cubes).

They compare this to walking to the door by repeatedly travelling half the remaining distance. They shade different representations of the fractions (as shown on the right) to create math art.

Students share the story *To Infinity & Beyond* and their learning at home.

Students in **grades 5-9** also compare the two fraction patterns on the right and demonstrate that 1/4 + 1/16 + 1/64 + … = 1/3 and 1/2 + 1/8 + 1/32 + … = 2/3. They also develop an understanding of 0.999… = 1.

ACTIVITY 1: What do you see?

1. What do you see?

WHAT STUDENTS MIGHT SEE

1. Shrinking patterns

Students may notice the shrinking patterns of pink, blue, yellow, green, and so forth. They may also notice that these patterns represent the fractions 1/2, 1/4, 1/8, 1/16, and so forth.

They may wonder how many of the fractions in this pattern would fit in a square, and what the smallest fraction might be.

2. Infinity in your hand

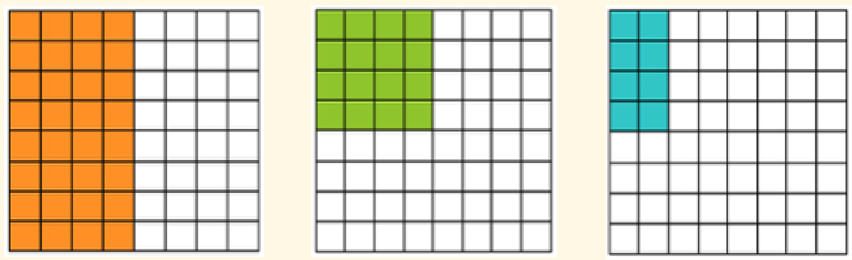
Students may propose that the infinite number of the fractions 1/2, 1/4, 1/8, 1/16, and so forth, would fit in a single square.

This implies that 1/2 + 1/4 + 1/8 + 1/16 + … = 1.

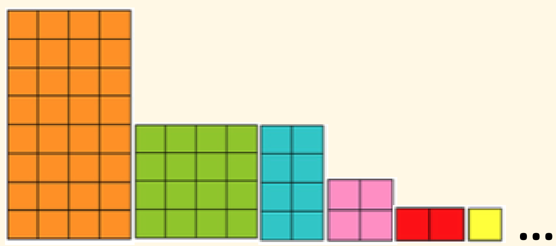
It also implies that they can hold infinity in their hand, which is further elaborated in Activity 2.

ACTIVITY 2: How big will it be?

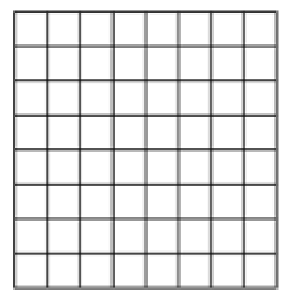
2. Imagine shading square grids in this way forever.



Then cutting out the shaded parts to form a new shape.



How big may the new shape become?



HANDS ON

Working in pairs, students use five 8x8 grids to shade and represent the fractions 1/2, 1/4, 1/8, 1/16, and 1/32. They then cut out the shaded parts and join them to form a new shape (as shown above).

We have worked in several classrooms, in grades 2 and up, where students completed this activity. Students quickly notice that these fractions will fit in a single square.

Many students are convinced that the sum 1/2 + 1/4 + 1/8 + 1/16 + … is 1. Others, who view the square as never completely filling, are convinced that the sum would not be more than 1.

WALKING TO THE DOOR

Referring to the image on the right, you may ask students if it is possible to walk to the door, and beyond, if they first walk halfway to the door, then half the remaining distance, then the remaining distance, and so on.

One teacher candidate noted: *Of course it is possible. Even if I take a single step and then look back, I can imagine that I travelled the infinite fractions 1/2, 1/4, 1/8, 1/16, and so forth. Infinity is everywhere.*

ACTIVITY 3: Create math art

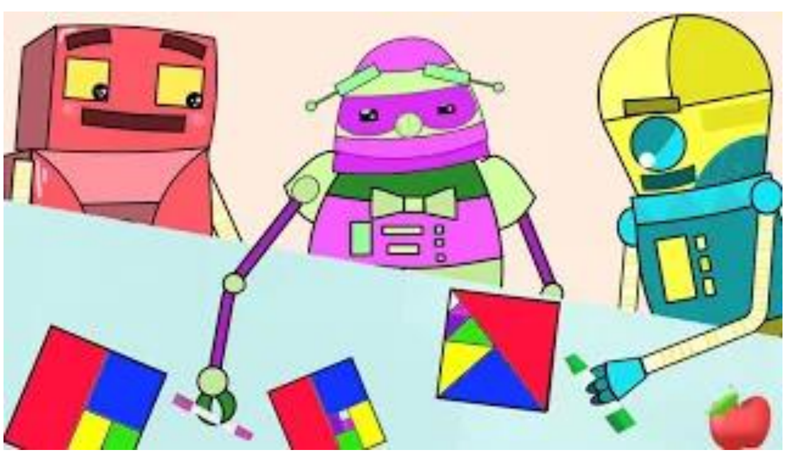
3. What math art can you create?

INFINITY ART

The images above are an opportunity for students to engage aesthetically with the concepts of infinity and area representations of fractions.

MATH ART

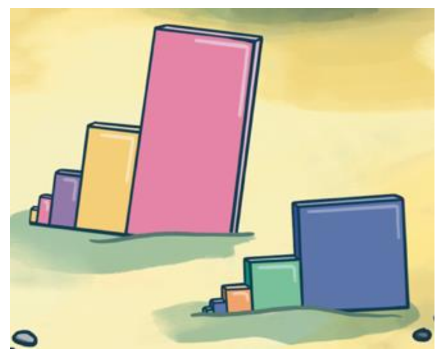
For more inspiration, view the *Math Art* video at https://youtu.be/yLte3H9mqmo



ACTIVITY 4: What does this represent?

4. What does this represent?

SOLUTION



In the 2 images on the right, we see that:

- the pink rectangle is twice the size of the blue square;
- the yellow rectangle is twice the size of the green square;
- the purple rectangle is twice the size of the orange square;
- and so forth.

So, the rectangular pieces make up 2/3 of the square below, and the remaining sqaure pieces make up the remaining 1/3.

That is, 1/2 + 1/8 + 1/32 + … = 2/3.

And 1/4 + 1/16 + 1/64 + … = 1/3.

0.999… = 1

Does 0.999… equal 1? Yes. Here is a proof.

1/3 = 0.333…
1/3 = 0.333…
<u>1/3 = 0.333…</u>
3/3 = 1 = 0.999…

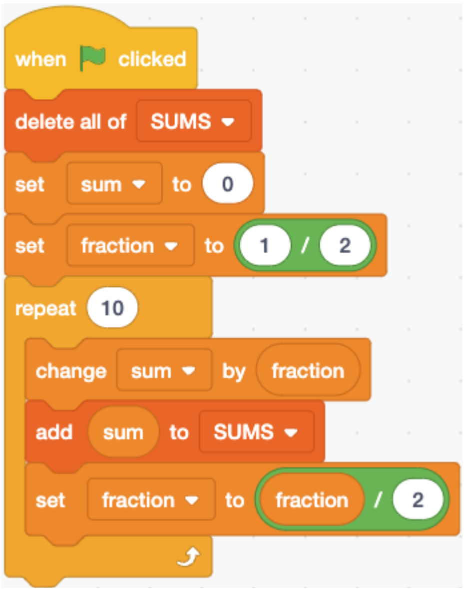
Or, using the fractions above:

2/3 = 0.666…
<u>1/3 = 0.333…</u>
3/3 = 1 = 0.999…

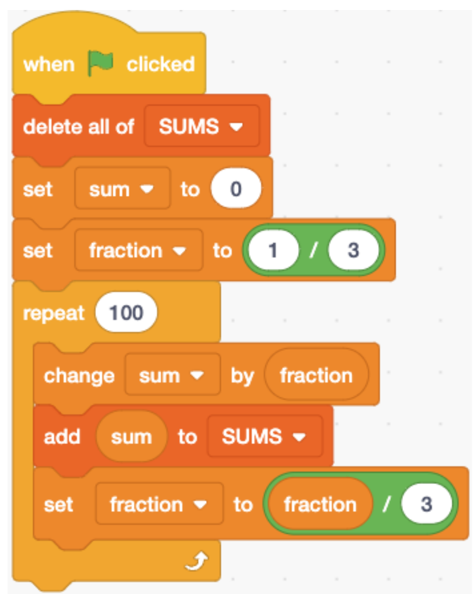
CODING PUZZLE 1: Infinity (Scratch)

Go to https://scratch.mit.edu/projects/964712868/editor .

Run the code to see the list of numbers shown below.



1. The code finds the sum of 10 fractions.
   a) Which fractions are they?
   b) What is the sum if this pattern continues forever?
2. Edit the code as shown on the right to find the sum of fractions 1/3, 1/9, 1/27, and so on. What is the sum?
3. Edit the code to find the sum of fractions 1/4, 1/16, 1/64, and so on. What is the sum?
4. Edit the code to find the sum of fractions 2/3, 2/9, 2/27, and so on.
   a) What is the sum?
   b) How does this sum make sense?



SOLUTIONS

1.a) 1/2, 1/4, 1/8, 1/16 … 1.b) 1 2. 1/2 3. 1/3

4.a) 1

4.b) Remember the walk to the door. We are always taking a fraction of the distance remaining. So, we are always getting closer to the door. The larger the fraction, like 9/10 vs 1/2, the closer we get to the door.

CODING PUZZLE 2: Infinity (Python)

Go to https://cscircles.cemc.uwaterloo.ca/console .

This Python code serves the same purpose as the Scratch code in the previous puzzle.

Enter and run this Python code to list the numbers shown on the right.

```
SUM = 0
fraction = 1/2
for x in range(1,11):
    SUM = SUM + fraction
    fraction = fraction/2
    print(SUM)
```

```
0.5
0.75
0.875
0.9375
0.96875
0.984375
0.9921875
0.99609375
0.998046875
0.9990234375
```

1. The code finds the sum of 10 fractions.
   a) Which fractions are they?
   b) What is the sum if this pattern continues forever?
2. Edit the code as shown on the right to find the sum of fractions 1/3, 1/9, 1/27, and so on. What is the sum?
3. Edit the code to find the sum of fractions 1/4, 1/16, 1/64, and so on. What is the sum?
4. Edit the code to find the sum of fractions 2/3, 2/9, 2/27, and so on.
   a) What is the sum?
   b) How does this sum make sense?

```
SUM = 0
fraction = 1/3
for x in range(1,11):
    SUM = SUM + fraction
    fraction = fraction/3
    print(SUM)
```

SOLUTIONS

1.a) 1/2, 1/4, 1/8, 1/16 … 1.b) 1

2. 1/2

3. 1/3

4.a) 1

4.b) Any fraction we use, from 1/2 to less than 1 (like 1/2, 2/3, 3/4, 9/10 …), would result in a sum of 1. Think back to the fractions walk to the door. We always walked a fraction of the distance left to travel. So, we are always getting closer to the door. The larger the fraction, like 9/10 vs 1/2, the faster we get closer to the door.

NATURAL DENSITY

Natural density, and infinity and limit, are in Grade 9 mathematics in Ontario.

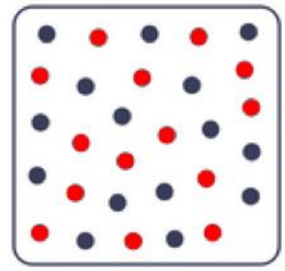
Natural numbers = {1, 2, 3, 4, 5, 6, 7, 8 …}

Even numbers = {2, 4, 6, 8, 10 …}

Natural density of even numbers

= Chance of picking an even number from the infinite set of natural numbers

= 0.5



PUZZLES

1. What is the natural density of the odd numbers: {1, 3, 5, 7, 9 ...}?
2. What is the natural density of multiples of 3: {3, 6, 9, 12, 15 ...}?
3. What is the natural density of multiples of 10: {10, 20, 30, 40, 50 ...}?
4. What is the natural density of the number 1?
5. What is the natural density of the square numbers: {1, 4, 9, 16, 25 ...}?

SOLUTIONS

1. 0. 5 or 1/2 (as half the natural numbers are odd)
2. 0.333... or 1/3 (since 1/3 of the natural numbers are multiples of 3)
3. 0.1 or 1/10 (since 1/10 of the natural numbers are multiples of 10)
4. 0

   For natural numbers 1-10, there is 1/10 chance of randomly picking the number 1.

   For natural numbers 1-100, there is 1/100 chance of randomly picking the number 1.

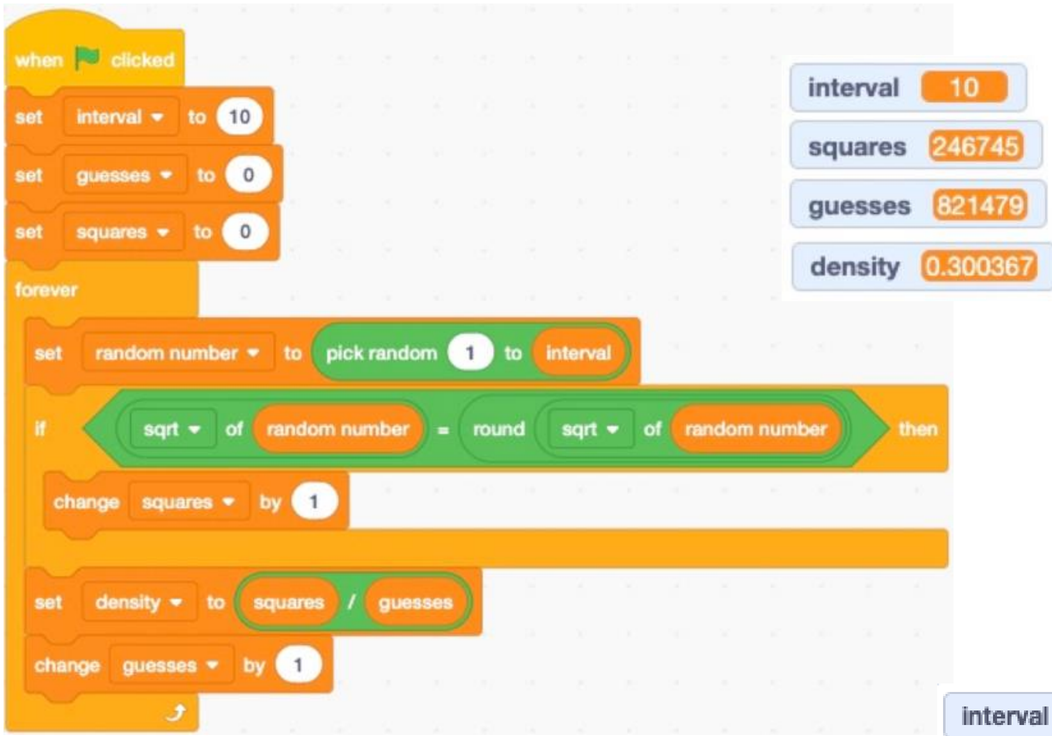
   For natural numbers 1-1000, there is 1/1000 chance of randomly picking the number 1.

   As we consider large and larger intervals, the probability decreases. Its limit is 0.

   This is like the walk to the door, where the distance left to walk is decreasing, and getting closer and closer to 0, which is its limit.

5. 0

- Run the code at https://scratch.mit.edu/projects/565845359/editor
- What does this code do?



- Run the code with different intervals: 100, 10,000 and 1,000,000.
- Notice how the density changes.
- What is the limit of the natural density values?

ABOUT THE NATURAL DENSITY OF SQUARE NUMBERS:

The natural density of square numbers is counter-intuitive, as there is an infinite number of square numbers. How can their density be 0?

But as we see from the coding simulation, the density of square numbers gets closer and closer to 0 as we consider larger intervals of natural numbers.

We may also look this problem algebraically:

Natural density of square numbers

= number of square numbers / number of natural numbers

= $\sqrt{N} / N = 1 / \sqrt{N}$

(Note: In the first N natural numbers, there are at most $\sqrt{N}$ square numbers)

As N becomes larger and larger, $1 / \sqrt{N}$ gets closer and closer zero.